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THE LOCAL MODULI PROBLEM APPLICATIONS
TO THE CLASSIFICATION OF
ISOLATED HYPERSURFACE SINGULARITIES

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Notations

k field

\mathbb{C} is the field of complex numbers

$k[\underline{x}] = k[x_1, \dots, x_n]$

k^{\times} the multiplicative group of k

\mathbb{P}^n the projective n -space

$\underline{\mathcal{A}}$ the category of local artinian k -algebras
with residue field k

$\underline{\mathcal{A}}_{H^{\wedge}}$ the category of local artinian H^{\wedge} -algebras
with residue field k

$\underline{\text{gr}}$ the category of groups

$\text{aut}_S(X \otimes_k S) = \{ \phi \in \text{Aut}_S(X \otimes_k S) \mid \phi \otimes k = 1 \}, S \text{ in } \underline{\mathcal{A}}$

$\text{aut}_R(X \otimes_{H^{\wedge}} R) = \{ \phi \in \text{Aut}_R(X \otimes_{H^{\wedge}} R) \mid \phi \otimes R = 1 \}, R \text{ in } \underline{\mathcal{A}}_{H^{\wedge}}$

§ 1. INTRODUCTION

The purpose of this paper is to contribute towards a classification of isolated hypersurface singularities, modulo the action of the contact group, see § 4.

We have tried to understand the behaviour of the Tjurina number τ in a μ -constant deformation, and to compute the maximal dimension of a nowhere constant family in the μ -constant region.

Our main results in this direction may be summarized as follows, see (4.5) and (4.6).

Let f be a quasihomogenous hypersurface with an isolated singularity at the origin, and let $F(t_1, \dots, t_{m_0})$ be the miniversal μ -constant deformation of f . Then there exist a finite collection of analytic families

$$\underline{X}_\tau \rightarrow \underline{M}_\tau, \quad 0 \leq \tau \leq \mu = \tau(f)$$

of hypersurface singularities with constant Tjurina number τ , containing as fibers all $F(t_1, \dots, t_{m_0})$ and such that, putting $\tau_{\min} = \min_{\underline{t}} \tau(F(\underline{t}))$,

- (1) \underline{M}_τ is of finite type
- (2) $\dim \underline{M}_\tau < \dim \underline{M}_{\tau_{\min}} = m_0 + \tau_{\min}^{-\mu}$
- (3) $\underline{M}_\tau \neq \emptyset$ for all τ with $\tau_{\min} \leq \tau \leq \mu$
- (4) If $\underline{D} \subseteq \underline{M}_\tau$ is a subscheme such that $\underline{X}_\tau|_{\underline{D}}$ is constant, then \underline{D} is a closed point.

Moreover we show how to compute the μ -modality of f , i.e. the number

$$\mu m(f) = \max_{\tau} \{ \dim \underline{M}_\tau \} = m_0 + \tau_{\min}^{-\mu}$$

and thereby τ_{\min} , see (4.15) and (4.18).

The first 3 paragraphs of this paper are concerned with the general framework in which we have chosen to put our results on hypersurface singularities.

Given a field k and some algebraic object X such as,

Example 1. $X = \underline{c}$, a small category of k -schemes. Put

$$A^i(k, \underline{c}, \underline{O}_{\underline{c}}) = A^i \quad i \geq 0, \text{ see [La].}$$

Example 2. $X = \text{Spec}(A)$, A any k -algebra with isolated singularities. In particular, we shall be interested in the case

where $A = (k[X_1, \dots, X_n]/(f))_{(X_1, \dots, X_n)}$ is the local ring of an isolated hypersurface singularity. In this case $A^i = H^i(k, A; A)$, $i \geq 0$ is the André cohomology.

Example 3. $X = \underline{e}$, a small category of O_Y -Modules where Y is some k -scheme. Here $A^i = \text{Ext}_{O_Y}^i(O_{\underline{e}}, O_{\underline{e}})$, $i \geq 0$ are defined as in [La] with Hom replacing Der . See the concluding remark, loc.cit. p. 150.

Example 4. $X = E$, a coherent $O_{\mathbb{P}^n}$ -Module. $A^i = \text{Ext}_{O_{\mathbb{P}^n}}^i(E, E)$, $i \geq 0$.

Of particular interest is the case where E is a bundle.

Assume now that $\dim_k A^i < \infty$ for $i = 1, 2$. Then, see [La], (4.2.4), there exist in all these cases a formal moduli H^\wedge (a prorepresenting hull for the deformation functor) of X , and a formal versal family

$$\pi^\wedge: X^\wedge \rightarrow \text{Spf}(H^\wedge) = \underline{H}^\wedge$$

Assume moreover, that there is an algebraization of π^\wedge , denote it by

$$\pi: \tilde{X} \rightarrow \text{Spec}(H) = \underline{H}.$$

The first part of this paper is devoted to the study of π in this generality.

In §2 we prove that there is always a formal prorepresenting sub-stratum \underline{H}_0^\wedge of \underline{H}^\wedge universal with respect to the property

$$\text{Mor}(H_0^\wedge, -) \rightarrow \text{Def}_X(-)$$

on the category of artinian local k -algebras with residue field k .

In §3 we construct the Kodaira-Spencer map $g_X: \theta_{\underline{H}} \rightarrow A^1(H, \tilde{X}; \mathcal{O}_{\tilde{X}})$.

Using the properties of g_X we prove, under rather strong assumptions $R1., R2., R3.$ and $R4.$, that there exists a finite collection of flat analytic families $(k=\mathbb{C})$,

$$\pi_{\tau,i}: \underline{X}_{\tau,i} \rightarrow \underline{M}_{\tau,i}$$

such that

- (1) $\underline{M}_{\tau,i}$ is connected, of finite type, locally isomorphic to the prorepresenting substratum of some $\underline{H}(x)$.
- (2) All deformations of X , close to X , occur among the fibers of $\pi_{\tau,i}$, $\tau = 0, \dots, \tau^1 = \dim_k A^1$, $i=1, \dots, r_\tau$.
- (3) $\pi_{\tau,i}$ is nowhere constant, i.e. if $\underline{D} \subseteq \underline{M}_{\tau,i}$ is a subscheme such that $\pi_{\tau,i}/\underline{D}$ is constant, then \underline{D} is a closed point.

This paper is a preliminary version. Some of the proofs are therefore cut down to the bare minimum. Details will occur in the final publication.

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§ 2. THE PROREPRESENTING SUBSTRATUM OF THE FORMAL MODULI

Let X be any algebraic object of the type discussed in the Introduction, and consider the deformation functor

$$\text{Def}_X: \underline{\mathcal{L}} \rightarrow \underline{\text{Sets}},$$

the corresponding cohomology $A^i = A^i(k, X; \mathcal{O}_X)$, $i \geq 0$ and the universal obstruction morphism

$$\mathcal{O}_X: T^2 \rightarrow T^1$$

where $T^i = \text{Sym}_k(A^{i*})^\wedge$.

Denote by

$$H^\wedge = T^1 \otimes_{T^2} k$$

the formal moduli of X , i.e. the prorepresentable hull of the deformation functor Def_X , and put

$$\underline{H}^\wedge = \text{Spf}(H^\wedge).$$

In general there are lots of infinitesimal automorphisms of X , and obstructions for lifting these (see [Sch]). Therefore H^\wedge does not necessarily prorepresent Def_X .

However, as we shall see there is a universal prorepresenting substratum \underline{H}_0^\wedge of \underline{H}^\wedge , corresponding to a quotient

$$H_0^\wedge = H^\wedge / \sigma$$

of H^\wedge .

In fact, let us consider the category $\underline{\mathcal{L}}_H$ of all artinian local H^\wedge -algebras with residue field k .

Let X^\wedge be the formal versal family on H^\wedge defined by the identity $1_{H^\wedge} \in \text{Mor}(H^\wedge, H^\wedge)$ and consider the functor

$$\underline{\text{aut}}_{X^\wedge}: \underline{\mathcal{L}}_H \rightarrow \underline{\text{gr}}$$

defined by:

$$\underline{\text{aut}}_{X^\wedge}(S) = \{ \phi \in \text{Aut}_S(X^\wedge \otimes_{H^\wedge} S) \mid \phi \otimes k = 1_S \} := \text{aut}_S(X^\wedge \otimes_{H^\wedge} \hat{S}).$$

Theorem (2.1). Assume $\dim_k A^i$ is countable $i = 0, 1$. Then there exists a morphism of complete local H^\wedge -algebras

$$\underline{o}_a: H \underset{k}{\otimes} T^1 \rightarrow H \underset{k}{\otimes} T^0$$

such that

$$a_{X^\wedge} = (H \underset{k}{\otimes} T^0) \underset{H \underset{k}{\otimes} T^1}{\otimes} H^\wedge$$

is a prorepresenting hull for the functor $\underline{aut}_{X^\wedge}$.

Proof. This follows from the proof of [La], (4.2.4) with $\underline{aut}_{X^\wedge}$ replacing Def_X and A^{i-1} replacing A^i , $i = 1, 2$. Q.E.D.

Recall that there is the usual automorphism functor of X ,

$$\underline{Aut}_X: \underline{sch}/k \rightarrow \underline{gr}$$

defined by:

$$\underline{Aut}_X(S) = \text{Aut}_S(X \times S)$$

Assume \underline{Aut}_X is represented by the k -scheme $\text{Aut}(X)$ and let $1 \in \text{Aut}(X)$ be the identity element. Then the completion $\hat{O}_{\text{Aut}(X), 1}$ of the local ring of $\text{Aut}(X)$ at 1 , represents the fiber-functor of \underline{Aut}_X at $1 \in \text{Aut}_k(X)$, i.e. the functor

$$\underline{aut}_X: \underline{sch} \rightarrow \underline{gr}$$

defined by

$$\underline{aut}_X(S) = \{\phi \in \text{Aut}_S(X \otimes_k S) \mid \phi \otimes_k 1 = 1\} := \text{aut}_S(X \otimes_k S)$$

Let a_X be the prorepresentable hull of \underline{aut}_X , such that with the assumption above

$$a_X \approx \hat{O}_{\text{Aut}(X), 1}.$$

Assume from now on that $\text{Aut}(X)$ is smooth, implying that

$$a_X \approx \text{Sym}_k(A^{0*})^\wedge \quad (\text{see [La] Ch. 4}).$$

Definition (2.2). Let the ideal $\mathcal{O} \subseteq H^\wedge$ be generated by the coefficients of the elements of $\mathcal{O}_{\underline{a}}(\underline{m}) \subset H^\wedge \otimes_k T^0$, \underline{m} being the maximal ideal of $H^\wedge \otimes T^1$.

Then the prorepresenting substratum

$$\underline{H}_0^\wedge \subseteq H^\wedge$$

is the formal sub-scheme defined by \mathcal{O} .

Put $H_0^\wedge = H^\wedge / \mathcal{O}$. Then $\underline{H}_0^\wedge = \text{Spf}(H_0^\wedge)$ and we shall, mildly abusing the notations, also speak about the prorepresenting substratum H_0^\wedge .

By construction of $\mathcal{O}_{\underline{a}}$ it is clear that H_0^\wedge is the maximal quotient of H^\wedge for which

$$a_{X^\wedge \otimes H^\wedge} H_0^\wedge$$

is H_0^\wedge -smooth.

Proposition (2.3). H_0^\wedge is the maximal quotient of H^\wedge for which the canonical morphism of functors on \underline{I} ,

$$\rho_0: \text{Mor}(H_0^\wedge, -) \rightarrow \text{Def}_X$$

is injective.

Proof. Let H_1^\wedge be a quotient of H^\wedge , and assume $\phi_1, \phi_2 \in \text{Mor}(H_1^\wedge, R)$ are mapped onto the same element $\bar{\phi}_1 = \bar{\phi}_2$ in $\text{Def}_X(R)$. This, of course, means that there exists an R -isomorphism $X^\wedge \otimes_{H^\wedge} R / \phi_1 \xrightarrow{\cong} X^\wedge \otimes_{H^\wedge} R / \phi_2$ where at the left side R is considered as H^\wedge -module via ϕ_1 and at the right hand side R is considered as H^\wedge -module via ϕ_2 .

We may assume, by induction, $\phi_1 \equiv \phi_2 \pmod{\underline{n}}$ where \underline{n} is some ideal of R killed by the maximal ideal \underline{m}_R .

Then $\bar{\phi} \otimes_R R/\underline{n}$ is an automorphism of $X^\wedge \otimes_{H^\wedge} R/\underline{n}$, corresponding to a morphism $a_{X^\wedge \otimes H_1^\wedge} \rightarrow R/\underline{n}$. If $a_{X^\wedge \otimes H_1^\wedge}$ is formally

H_1 -smooth, then obviously this morphism may be lifted to a morphism $a_{X^\wedge \otimes H_1^\wedge} \rightarrow R$, proving that $\Phi \otimes_R R/\underline{n}$ is liftable as an automorphism to some $\Phi_1: X^\wedge \otimes_{H^\wedge} R|_{\phi_2} \approx X^\wedge \otimes_{H^\wedge} R|_{\phi_2}$. But then $\Phi \circ \Phi_1^{-1}: X^\wedge \otimes_{H^\wedge} R|_{\phi_1} \xrightarrow{\sim} X^\wedge \otimes_{H^\wedge} R|_{\phi_2}$ is an isomorphism extending the identity of $X^\wedge \otimes_{H^\wedge} R/\underline{n}$. Thus $\phi_1 = \phi_2$. From this follows that $\rho_0: \text{Mor}(H_0^\wedge, -) \rightarrow \text{Def}_X$ is injective.

Conversally assume H_1^\wedge is a quotient of H^\wedge such that $\rho_1: \text{Mor}(H_1^\wedge, -) \rightarrow \text{Def}_X$ is injective. If R is any quotient of H_1^\wedge , then any automorphism $\Phi_{\underline{n}}$ of $X^\wedge \otimes_{H^\wedge} R/\underline{n} = (X^\wedge \otimes_{H^\wedge} H_1^\wedge) \otimes_{H_1^\wedge} R/\underline{n}$ may always be lifted to an automorphism of $X^\wedge \otimes_{H^\wedge} R$. It follows that $a_{X^\wedge \otimes_{H^\wedge} H_1^\wedge}$ has to be formally smooth, which proves the proposition. Q.E.D.

Remark (2.4). Recall that H/\underline{m}^2 represents the restriction of the deformation functor Def_X to the subcategory $\underline{\ell}_2 = \{R \in \underline{\ell} \mid \underline{m}_R^2 = 0\}$ of $\underline{\ell}$. Notice that, never the less, H/\underline{m}^2 is rarely a quotient of H_0^\wedge .

Consider for any $n \geq 0$ the subfunctors Def_X^n of Def_X defined by:

$$\text{Def}_X^n(R) = \{X_R \in \text{Def}_X(R) \mid A^n(R, X_R; O_{X_R}) \text{ is a deformation of } A^n(k, X; O_X)\}$$

Then one may prove that Def_X^n has a prorepresentable hull H_n^\wedge which is a quotient of H^\wedge , defining the n-th equicohomological substratum H_n^\wedge of the formal moduli H^\wedge , see [St].

Proposition (2.5). Suppose k algebraically closed and suppose moreover that for every R in $\underline{\ell}_{H^\wedge}$ the group scheme $\underline{\text{Aut}}_R(X^\wedge \otimes_{H^\wedge} R)$ defined by $\underline{\text{Aut}}_r(X^\wedge \otimes_{H^\wedge} R)(T) = \underline{\text{Aut}}_{R \otimes_k T}(X^\wedge \otimes_{H^\wedge} R \otimes_k T)$ is algebraic and k -smooth. Then the prorepresenting substratum coincides with the 0-th equicohomological substratum.

Proof. Let $R \rightarrow S$ be a morphism in $\frac{\ell}{H}^{\wedge}$, and consider the induced maps

- (i) $\text{Aut}_R(X^{\wedge}_{H^{\wedge}} \otimes R) \rightarrow \text{Aut}_S(X^{\wedge}_{H^{\wedge}} \otimes S)$
- (ii) $A^0(R, X^{\wedge}_{H^{\wedge}} \otimes R; O_{X^{\wedge} \otimes R}) \rightarrow A^0(S, X^{\wedge}_{H^{\wedge}} \otimes S; O_{X^{\wedge} \otimes S})$

Since for every R in $\frac{\ell}{H}^{\wedge}$, $\text{Lie}(\text{Aut}_R(X^{\wedge}_{H^{\wedge}} \otimes R)) = A^0(R, X^{\wedge}_{H^{\wedge}} \otimes R; O_{X^{\wedge} \otimes R})$, we are through by ([D.-G.], II, §5 (5.3)).

Q.E.D.

Remark (2.6). If X is a linear map $\phi: k^m \rightarrow k^n$, then as one easily checks, the formal moduli is algebraic and is given by $\underline{H} = \underline{A}^{m \cdot n} = \text{Hom}_k(k^m, k^n)$. The prorepresenting substratum $\underline{H}_0 \subseteq \underline{H}$ is then called the bifurcation diagram (see f.eks. Arnold: [A]) of $\phi \in \underline{H}$, and consists of the points of \underline{H} , in the neighbourhood of o , for which the corresponding matrix has the same Jordan blocks as ϕ .

Remark (2.7). Let \underline{H}_i^{\wedge} be the i -th equicohomological substratum of \underline{H}^{\wedge} , and assume that the formal family $X_i^{\wedge} \rightarrow \underline{H}_i^{\wedge}$ is the completion of an algebraic family $\tilde{X}_i \rightarrow \underline{H}_i$. Let \underline{H}_{∞} be the intersection of the \underline{H}_i 's. Then $A^i(H_{\infty}, \tilde{X}_{\infty}; O_{\tilde{X}_{\infty}})$ is H_{∞} -flat. Suppose that $A^i(H_{\infty}, \tilde{X}_{\infty}; O_{\tilde{X}_{\infty}})$ is of finite type over H_{∞} . Then, in particular,

$A^i(H_{\infty}/\underline{m}_x^n; \tilde{X}_{\infty} \otimes_{H/\underline{m}_x^n} O_{\tilde{X}_{\infty}} \otimes_{X_{\infty}^{\wedge} H/\underline{m}_x^n} H/\underline{m}_x^n)$ is reflexive as an H/\underline{m}_x^n -module, for all $i \geq 0$ and all $x \in \underline{H}_{\infty}$.

Now assuming we have a flat family $\tilde{X} \rightarrow \text{Spec}(S)$ such that $A^i(S, \tilde{X}; O_{\tilde{X}})$ is reflexive as an S -module for $i = 1, 2$, there exist a morphism of complete S -algebras

$$T_S^2 = \text{Sym}_S(A^{2*})^\wedge \rightarrow T_S^1 = \text{Sym}_S(A^{1*})^\wedge$$

such that the S-algebra

$$(T_S^1 \hat{\otimes}_{T_S^2} S) \otimes_S \underline{k}(s)$$

is the formal moduli of \tilde{X}_S for all $s \in \text{Spec}(S)$. Under the assumptions above, there exists a morphism

$$H_\infty \rightarrow T_{H_\infty}^1 \hat{\otimes}_{T_{H_\infty}^2} H_\infty$$

for which $(T_{H_\infty}^1 \hat{\otimes}_{T_{H_\infty}^2} H_\infty) \otimes_{H_\infty} \underline{k}(x) \simeq H(x)$ for every $x \in \underline{H}_\infty$.

(The proof of this parallels the proof of [La;(4.4.2)].)

§ 3. THE KODAIRA - SPENCER MAP AND THE FUNDAMENTAL DISTRIBUTION

Let S be any k -algebra and consider a flat family

$$\pi: \tilde{X} \rightarrow \text{Spec}(S).$$

Corresponding to the simplicial k -algebra

$$\begin{array}{c} v_1 \\ \rightarrow \\ S \rightleftarrows S \otimes_k S \rightleftarrows S \otimes_k S \cdots \cdots \\ \leftarrow \\ v_2 \end{array} \quad , \quad v_1 = \text{id} \otimes 1, \quad v_2 = 1 \otimes \text{id}$$

one may define a series of obstructions for descent of \tilde{X} to k .

The first descent obstruction is gotten in the following way.

Put $I = \ker\{S \otimes_k S \xrightarrow{m} S\}$, where m is the multiplication, and consider the diagram

$$\begin{array}{c} v_1 \\ \rightarrow \\ 0 \leftarrow S \rightleftarrows S \otimes_k S/I^2 \leftarrow I/I^2 \leftarrow 0 \\ \leftarrow \\ v_2 \end{array} \quad , \quad I/I^2 \simeq \Omega_{S/k}.$$

Since $v_1^*(\tilde{X})$ and $v_2^*(\tilde{X})$ are two liftings of \tilde{X} to $S \otimes_k S/I^2$, the difference

$$g(\tilde{X}) = v_1^*(\tilde{X}) - v_2^*(\tilde{X})$$

sits in $A^1(S, \tilde{X}; \mathcal{O}_{\tilde{X}} \otimes_S \Omega_{S/k})$. Suppose S is k -smooth, then $g(\tilde{X}) \in A^1(S, \tilde{X}; \mathcal{O}_{\tilde{X}} \otimes_S \Omega_{S/k})$ defines a morphism

$$g: \theta_{S/k} \rightarrow A^1(S, \tilde{X}; \mathcal{O}_{\tilde{X}})$$

Proposition (3.1). Suppose S is smooth, and let $s \in \text{Spec}(S)$ be a k -point. Then the induced map

$$g_s = g \otimes_{S/k} k(s): \theta_{S,s} \rightarrow A^1(\underline{k}(s), X(s); \mathcal{O}_{X(s)})$$

where $X(s) = \pi^{-1}(s)$, is the tangent map of the canonical morphism $S_s^\wedge \rightarrow H^\wedge(X(s))$ defined by the formal deformation $\hat{X} \otimes_S S_s^\wedge$ of $X(s)$ to S_s^\wedge . (Here $H^\wedge(X(s))$ is the formal moduli of $X(s)$.)

Proof. Let $P_i = S \otimes_k S/I^2$, $i = 1, 2$ be the ring $S \otimes_k S/I^2$ considered as left S -algebra via v_1 and v_2 respectively, and as right S -algebra via v_2 . Then, as left S -algebra we have the following isomorphisms

$$P_1 \otimes_S \underline{k}(s) \simeq S/\underline{m}_S^2$$

$$P_2 \otimes_S \underline{k}(s) \simeq S/\underline{m}_S^2 = \underline{k}(s) [\underline{m}_S/\underline{m}_S^2]$$

By definition, we have

$$v_i^*(\tilde{X}) = \tilde{X} \otimes_S P_i, \quad i = 1, 2$$

therefore,

$$v_i^*(\tilde{X}) \otimes_S \underline{k}(s) = \begin{cases} X(s) \otimes_{\underline{k}(s)} \underline{k}(s) [\underline{m}_S/\underline{m}_S^2] & i = 2 \\ \tilde{X} \otimes_S S/\underline{m}_S^2 & i = 1 \end{cases}$$

Consider the exact sequence of right S -modules

$$0 \leftarrow S \leftarrow S \otimes_k S/I^2 \leftarrow \Omega_{S/k} \leftarrow 0$$

tensorise with $\underline{k}(s)$ over S , and obtain the 3-dimensional commutative diagram

$$\begin{array}{ccccc} & & \Omega_{S/k} & & \\ & \swarrow & \downarrow & \searrow & \\ & S \otimes_k S/I^2 & \underline{m}_S/\underline{m}_S^2 & \tilde{X} \otimes_S P_1 & \\ & \downarrow & \downarrow & \downarrow & \\ S & \xleftarrow{\quad} \tilde{X} & \xrightarrow{\quad} \tilde{X} \otimes_S P_2 & & \\ \downarrow & \swarrow & \downarrow & \searrow & \\ \underline{k}(s) & \xleftarrow{\quad} X(s) & \xrightarrow{\quad} X(s) \otimes_{\underline{k}(s)} S/\underline{m}_S^2 & & \end{array}$$

Obviously $g(\tilde{X}) = \tilde{X} \otimes_S P_1 - \tilde{X} \otimes_S P_2 \in A^1(S, \tilde{X}; \Omega_{\tilde{X}} \otimes_S \Omega_{S/k})$ is, under the specialization map $S \rightarrow \underline{k}(s)$, mapped to $\tilde{X} \otimes_S S/\underline{m}_S^2 - X(s) \otimes_{\underline{k}(s)} S/\underline{m}_S^2 \in A^1(\underline{k}(s), X(s); \Omega_{X(s)} \otimes_{\underline{k}(s)} \underline{m}_S/\underline{m}_S^2)$.

Therefore $g \otimes \underline{k}(s)$ coincides with the map

$$\theta_{S, X} = (\underline{m}_S/\underline{m}_S^2)^* \rightarrow A^1(\underline{k}(s), X(s), \Omega_{X(s)})$$

defined by $\tilde{X} \otimes_S S/\underline{m}_S^2$.

Q.E.D.

Definition (3.2). The morphism of S -modules

$$g: \theta_{S/k} \rightarrow A^1(S, \tilde{X}; \mathcal{O}_{\tilde{X}})$$

is called the Kodaira-Spencer map.

The kernel of the Kodaira-Spencer map is a distribution on $\text{Spec}(S)$.

Denote it by \underline{V}_π or by \underline{V}_S when there is no risk of confusion.

Definition (3.3). $\underline{V}_\pi \subseteq \theta_S$ is called the Kodaira-Spencer distribution.

Suppose from now on, that,

- R1. $A^2(k, X; \mathcal{O}_X) = 0$
 then H^\wedge is non-singular. Suppose moreover that
- R2. there exists an algebraization

$$\pi: \tilde{X} \rightarrow \underline{H} = \text{Spec}(H)$$

of the formal versal family $\pi^\wedge: X^\wedge \rightarrow \underline{H}^\wedge$. And suppose finally that

- R3. $A^i(H, \tilde{X}; \mathcal{O}_{\tilde{X}})$ is an H -module of finite type, $i > 0$.

Lemma (3.4). Let $\tilde{Y} \rightarrow \text{Spec}(S)$ be any flat family, such that S is regular, and let \underline{m} be a maximal ideal of S such that $S/\underline{m} = k$. Assume $A^i(S, \tilde{Y}; \mathcal{O}_{\tilde{Y}})$ is an S -module of finite type for $i > 0$, then there exists a spectral sequence with

$$E_2^{p,q} = \text{Tor}_{-p}^S(A^q(S, \tilde{Y}; \mathcal{O}_{\tilde{Y}}), S/\underline{m}^n)$$

converging to

$$A^{p+q}(S/\underline{m}^n, \tilde{Y} \otimes_S S/\underline{m}^n; \mathcal{O}_{\tilde{Y} \otimes_S S/\underline{m}^n}).$$

Moreover,

$$A^i(S, \tilde{Y}; \mathcal{O}_{\tilde{Y}})^\wedge \simeq \varprojlim_n A^i(S/\underline{m}^n; \tilde{Y} \otimes_S S/\underline{m}^n; \mathcal{O}_{\tilde{Y} \otimes_S S/\underline{m}^n})$$

Proof. Suppose first that $\tilde{Y} = \text{Spec}(\tilde{A})$. Recall the definition of A^q , i.e.

$$A^q(S, \tilde{A}; \tilde{A}) = H^q(C^\bullet(S\text{-free}/\tilde{A}; \text{Der}_S(-, \tilde{A})) \quad (\text{see [La]}).$$

Let L^n be an S -free resolution of S/\underline{m}^n . We may assume L^n is finite. Consider the double complex

$$\begin{array}{c} C^\bullet(S\text{-free}/\tilde{A}; \text{Der}_S(-, \tilde{A})) \otimes_{S L^n} \\ \parallel \\ C^\bullet(S\text{-free}/\tilde{A}; \text{Der}_S(-, \tilde{A} \otimes_{S L^n})) \end{array}$$

Since \tilde{A} is S -flat, the first part of the lemma follows from the usual pair of spectral sequences. The second part follows by taking limits of the spectral sequence, noticing that $\lim_n (i) = 0$ for $i > 1$ since the projective systems involved are composed of finite dimensional k -vector spaces. Finally, the globalization presents no problem, (see [La] Chapter 3). Q.E.D.

Notice that (3.4) together with R1., R2. and R3. implies that for all $x \in \underline{H}$ in a neighbourhood of 0 we will have $A^2(k, X(x); \mathcal{O}_{X(x)}) = 0$ where $X(x) = \pi^{-1}(x)$. This in turn implies that

$$A^1(H, \tilde{X}; \mathcal{O}_{\tilde{Y}}) \otimes_{H} k(x) = A^1(k, X(x); \mathcal{O}_{X(x)}).$$

Now, consider the Kodaira-Spencer map

$$g = g_X: \theta_{\underline{H}} \rightarrow A^1(H, \tilde{X}; \mathcal{O}_{\tilde{X}}).$$

What we have done above implies that g_X is locally surjective. Restricting \underline{H} we may assume g_X surjective and that

$$\pi: \tilde{X} \rightarrow \underline{H}$$

is versal.

The corresponding Kodaira-Spencer distribution $\underline{V} = \underline{V}_{\underline{H}}$ will be referred to as the fundamental distribution.

Proposition (3.5). Let \underline{H}_0 be the subscheme of \underline{H} defined by $\underline{V} = 0$. Then for every point $x \in \underline{H}_0$, the formalization $\underline{H}_{0x}^\wedge$ of \underline{H}_0 at x is the prorepresenting substratum of the formal moduli $\underline{H}_x^\wedge = \underline{H}^\wedge(X(x))$ of $X(x)$.

Proof. Since g_X is surjective it follows that $\underline{V} = 0$ defines the 0-th equicohomological substratum of \underline{H} , therefore of each \underline{H}_x^\wedge . Therefore (3.5) follows from (2.5). Q.E.D.

R4. Assume from now on that for every $x \in \underline{H}$, $X(x)$ satisfies R1., R2., R3., and the conclusion of (2.5).

Consider the H -module $A^1(H, \tilde{X}; \mathcal{O}_{\tilde{X}})$, and let $\{S_\tau\}$, $\tau = 0, \dots, \tau^1 = \dim_k A^1(k, X; \mathcal{O}_X)$ be the flattening stratification of $A^1(H, \tilde{X}; \mathcal{O}_{\tilde{X}})$, (see [M], Lecture 8), and let $S_\tau = \bigcup_{i=1}^k S_{\tau,i}$ be the decomposition of S_τ in its connected components.

Notice that $S_{\tau,i}$ are locally closed subschemes of \underline{H} and that $S_{\tau^1} = \underline{H}_0$.

Put for every $x \in \underline{H}$, with $x \in S_{\tau,i}$

$$T_x = \bar{S}_{\tau,i}$$

Consider also, for any k -point $x \in \underline{H}$, the formal family

$$\pi_x^\wedge: \tilde{X}_x^\wedge \rightarrow \underline{H}_x^\wedge$$

where $\underline{H}_x^\wedge = \text{Spf}(H_{\underline{m}_x}^\wedge)$, \underline{m}_x being the maximal ideal of H corresponding to x .

As above we denote by $X(x)$ the fiber $\pi^{-1}(x)$. Obviously the closed fiber of π_x^\wedge is precisely $X(x)$.

Denote by $\underline{H}^\wedge(x)$ the formal moduli of $X(x)$, i.e. $\underline{H}^\wedge(x) = \text{Spf}(H^\wedge(x))$, and let

$$\pi(x): \tilde{X}(x) \rightarrow \underline{H}(x),$$

with $\underline{H}(x) = \text{Spec}(H(x))$ be an algebraization of the formal versal

family

$$\pi^{\wedge}(x): X^{\wedge}(x) \rightarrow \underline{H}^{\wedge}(x).$$

Then, by formal versality, there is a smooth morphism

$$\rho_x^{\wedge}: \underline{H}_x^{\wedge} \rightarrow \underline{H}^{\wedge}(x)$$

such that π_x^{\wedge} is the pullback of $\pi^{\wedge}(x)$ by ρ_x^{\wedge} .

M. Artins approximation theorem, (see [Ar]), implies the existence of an étale neighbourhood

$$\eta(x): \underline{E}(x) \rightarrow \underline{H}$$

of $x \in \underline{H}$, and a dominant morphism

$$\rho(x): \underline{E}(x) \rightarrow \underline{H}(x)$$

such that $\eta(x)^*(\pi) = \rho(x)^*(\pi(x))$.

Notice that, in particular, this implies that

$$\rho(x)^{-1}(\underline{H}_0(x)) = \eta(x)^{-1}(\underline{T}_x)$$

where $\underline{H}_0(x)$ is the prorepresenting stratum of $\underline{H}(x)$.

Put $\underline{T}(x) = \eta(x)^{-1}(\underline{T}_x)$, and let $\underline{V}(x)$ be the restriction of $\eta(x)^*(\underline{V})$ to $\underline{T}(x)$. $\underline{V}(x)$ is a distribution on $\underline{T}(x)$. The fibers of the restriction of $\rho(x)$ to $\underline{T}(x)$ are maximal integral submanifolds of $\underline{V}(x)$. Since $\rho(x)$ is smooth, it follows that these integral submanifolds are all smooth.

Put $\underline{D}(x) = \rho(x)^{-1}(0)$.

Proposition (3.6). Through every point $x \in \underline{H}$ there passes a smooth maximal integral submanifold \underline{D}_x for \underline{V} . Moreover

$$\underline{D}_x \subseteq \underline{T}_x \quad \text{and}$$

$$\dim \underline{H}_0(x) = \dim \underline{T}_x - \dim \underline{D}_x.$$

Proof. Let $\underline{D}_x = \eta(x)(\underline{D}(x))$ and glue.

Q.E.D.

The next result is the main result of this §.

Proposition (3.7). Assume $k = \underline{C}$, and let $0 \leq \tau \leq \tau^1$, $1 \leq i \leq r_\tau$. Then there is a unique way of glueing together the families

$$\tilde{X}_0(x) \rightarrow \underline{H}_0(x), \quad x \in S_{\tau,i}$$

in the category of analytic spaces, to form an analytic family

$$\pi_{\tau,i}: \underline{X}_{\tau,i} \rightarrow \underline{M}_{\tau,i}$$

such that,

- (i) $\underline{M}_{\tau,i}$ is a quotient of $S_{\tau,i}$
- (ii) all fibers $X(x)$ of π with $x \in S_{\pi,i}$ occur as fibers of $\pi_{\tau,i}$.

Proof. Since, locally, $\underline{H}(x)$ is a quotient of $\underline{E}(x)$, it is clear that $\underline{H}_0(x)$ is a quotient of $\underline{T}(x)$ by $\underline{V}(x)$. The rest is obvious.
Q.E.D.

Definition (3.8). The strict modality of X is the integer

$$sm(X) = \dim \underline{H}_0.$$

By the τ -modality of X we shall mean the integer

$$\tau m(X) = \max \{ \dim \underline{M}_{\tau,i} \mid i=1, \dots, t_\tau \}$$

Finally, the modality of X is defined to be the integer

$$m(X) = \max \{ \tau m(X) \mid 0 \leq \tau \leq \tau^1 \}$$

Notice that by construction,

$$sm(X(x)) = \dim \underline{T}_x - \dim \underline{D}_x$$

$$\tau m(X) = \max \{ sm(X(x)) \mid x \in S_\tau \}.$$

§ 4 APPLICATIONS TO ISOLATED SINGULARITIES OF HYPERSURFACES

From now on X is going to be a hypersurface f , i.e. $X = \text{spec}(A)$, $A = k[x_1, \dots, x_n]/(f)$ where $f = f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$. Being a very special complete intersection X satisfies all conditions $R1.$, $R2.$, $R3.$ and $R4.$ of § 3 except possibly the conclusion of (2.5). Moreover

$$A^1(k, X; \mathcal{O}_X) = k[x_1, \dots, x_n] / (f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}).$$

Pick a base for $A^1(k, X; \mathcal{O}_X)$ represented by $\{\lambda_i\}_{i=0}^{\tau^1-1}$, $\lambda_i \in k[\underline{x}]$, where as above,

$$\tau^1 = \dim_k A^1(k, X; \mathcal{O}_X).$$

Put, $H = k[t_0, \dots, t_{\tau^1-1}]$, $F = f + \sum \lambda_i t_i \in H[\underline{x}]$ and $\tilde{A} = H[\underline{x}]/(F)$. Then $\tilde{X} = \text{Spec}(\tilde{A}) \rightarrow \underline{H} = \text{Spec}(H)$ is the versal family with which we shall have to work.

Proposition (4.1). The Kodaira-Spencer map

$$g: \theta_{\underline{H}} \rightarrow A^1(H, \tilde{A}; \tilde{A})$$

is given by

$$g\left(\frac{\partial}{\partial t_i}\right) = \text{class of } \frac{\partial F}{\partial t_i} \text{ in } H[x_1, \dots, x_n] / (F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}).$$

Proof. By definition of g , it is the obstruction for lifting $1_{\tilde{A}} \in \text{Aut}_{\tilde{H}}(\tilde{A})$ to an isomorphism $\phi: i_0^*(\tilde{A}) \rightarrow i_1^*(\tilde{A})$ where $i_k: H \rightarrow \text{HoH}/I^2$, $k = 1, 2$ and $I = \ker\{\text{HoH} \rightarrow H\}$ are defined as the v_k 's in § 3.

Now $\text{HoH}/I^2 = k[t, u]/(t_i - u_i)^2$, $i_0^*(\tilde{A}) = k[\underline{t}, \underline{u}][\underline{x}]/F_t$, $i_1^*(\tilde{A}) = k[\underline{t}, \underline{u}][\underline{x}]/F_u$, where $F_t = f + \sum \lambda_i t_i$, $F_u = f + \sum \lambda_i u_i$. Obviously this obstruction is simply given by the difference $(F_t - F_u) = \sum \lambda_i (t_i - u_i)$ in $A^1(H, \tilde{A}; \tilde{A} \otimes I/I^2) = H[\underline{x}] / (F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}) \otimes_{\tilde{H}} I/I^2$. But then

$$g = \sum (\text{class of } \lambda_i) \cdot dt_i.$$

Q.E.D

Now, in this § we shall be interested in the deformations of the local ring $A_{(\underline{x})} = (k[\underline{x}]/(f))_{(\underline{x})}$, when f is a hypersurface with an isolated singularity at $\underline{x} = (0, \dots, 0)$. This local ring will be referred to as the isolated singularity f . It is not difficult to see that $A_{(\underline{x})}$ satisfies all the conditions $R1, \dots, R4$, the conclusion of (2.5) included.

Clearly $A^1(k, A_{(\underline{x})}; A_{(\underline{x})}) \simeq A^1(k, A; A)_{(\underline{x})}$ is generated by monomials. We may therefore pick a basis for $A^1(k, A_{(\underline{x})}; A_{(\underline{x})})$ represented by $\{\underline{x}^{\underline{\alpha}_i}\}_{i=0}^{\tau-1}$, where $\underline{\alpha}_0 = (0, \dots, 0)$ and,

$$\tau = \dim_k A^1(k, A_{(\underline{x})}; A_{(\underline{x})}) = \dim_k (k[\underline{x}] / (f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}))_{(\underline{x})}$$

is the Tjurina number of f .

To simplify notations, put $A. = A_{(\underline{x})}$.

Consider the quotient $H. = H/(t_0)$. Then

$$H. = k[t_1, \dots, t_{\tau-1}]$$

and $\tilde{X} \rightarrow \underline{H}$ induces a flat family

$$\tilde{X}. = \text{Spec}((H.[\underline{x}]/(F.))_S) \rightarrow \underline{H}. = \text{Spec}(H.)$$

where $F. = f + \sum_{i=1}^{\tau-1} \underline{x}^{\underline{\alpha}_i} t_i$, and S is the multiplicatively closed subset $\{k^* + (x_1, \dots, x_n)\}$ of $\tilde{A} = H.[\underline{x}]/(F.)$. We contend that $\tilde{X}. \rightarrow \underline{H}. is versal in the category of local k -algebras. In fact, one checks that$

$$\tilde{X}. \rightarrow \underline{H}.$$

is a flat family of local rings, corresponding to the localization of \tilde{X} along the section of \tilde{X} on $\underline{H}. \subseteq \underline{H}$, defined by putting $x_i = 0, i = 1, \dots, n$.

To show that it is versal one proceeds as in § 3 knowing, as we do, that

$$\underline{X}^\wedge \rightarrow \underline{H}^\wedge$$

is the formal versal family of $A(\underline{x})$ admitting a section.

Recall the definition of the Milnor number of an isolated hypersurface singularity, $\mu(f) = \dim_k \left(k[[x_1, \dots, x_n]] / \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \right)$.

Let $\underline{H}_\mu \subseteq \underline{H}$ be the closed substratum of \underline{H} where μ is kept constant and equal to $\mu = \mu(f)$.

Proposition (4.2). For all isolated singularities $f \in k[x_1, \dots, x_n]$,

$$\underline{H}_0 \subseteq \underline{H}_\mu.$$

Proof: Consider $N^1(H_0, (H_0[\underline{x}]/F_0)_S; (H_0[\underline{x}]/F_0)_S) = H^1_{H_0}$, where F_0 is the restriction of F to H_0 . By assumption $H^1_{H_0}$ is H_0 flat

of finite rank r^1 , lifting $H^1(k, A; A)$

Moreover in $H^1_{H_0} = H_0[\underline{x}] / (F_0, \frac{\partial F_0}{\partial x_1}, \dots, \frac{\partial F_0}{\partial x_n})_S$ the ideal $(\underline{x})^N$ must be zero for all N greater or equal to some N_0 .

Let $R = H_0/\underline{m}^{k+1} \rightarrow S = H_0/\underline{m}^k$ be the obvious surjection, let F_R and F_S be the corresponding restrictions of F , and consider the commutative diagram

$$\begin{array}{ccccccc}
 0 & & 0 & & 0 & & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow \text{Ann}(F_R)_N \otimes (\underline{m}^k/\underline{m}^{k+1}) \rightarrow k[[\underline{x}]]_N / (\frac{\partial f}{\partial x_i}) \otimes (\underline{m}^k/\underline{m}^{k+1}) \xrightarrow{f} k[[\underline{x}]]_N / (\frac{\partial f}{\partial x_i}) \otimes (\underline{m}^k/\underline{m}^{k+1}) \rightarrow H^1 \rightarrow 0 \\
 \downarrow j & & \downarrow \ell & & \downarrow \ell & & \downarrow \\
 0 \rightarrow \text{Ann}(F_R)_N \rightarrow R[[\underline{x}]]_N / (\frac{\partial F_R}{\partial x_i}) \xrightarrow{F_R} R[[\underline{x}]]_N / (\frac{\partial F_R}{\partial x_i}) \rightarrow H^1_R \rightarrow 0 \\
 \downarrow \rho & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow \text{Ann}(F_S)_N \rightarrow S[[\underline{x}]]_N / (\frac{\partial F_S}{\partial x_i}) \xrightarrow{F_S} S[[\underline{x}]]_N / (\frac{\partial F_S}{\partial x_i}) \rightarrow H^1_S \rightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0 & & 0
 \end{array}$$

where $R[[\underline{x}]]_N = R[\underline{x}]/(\underline{x})^N$ for $N_0 \leq N < \infty$

and $R[[\underline{x}]]_\infty = R[[\underline{x}]]$ (for $N = \infty$). Since $\{\frac{\partial F_R}{\partial x_i}\}_{i=1, \dots, n}$ is a regular sequence in $R[[\underline{x}]]$ all sequences in the diagram above are exact for $N = \infty$.

Suppose that $N < \infty$, then since $\dim_k H^1_R = \dim_k \text{Ann}(F_R)_N$ whenever $N \geq N_0$, it follows that the terms in the left vertical sequence are constant for $N_0 \leq N$, thus equal to the terms of the same sequence for $N = \infty$. Consequently j is injective for all $N \geq N_0$.

By induction we shall show that $R[[\underline{x}]]_N/(\frac{\partial F_0}{\partial x_i})$ is R -flat. Having shown this for all $R = H_0/\underline{m}^{\ell+1}$ it follows that $H_0[[\underline{x}]]/(\frac{\partial F_0}{\partial x_i})$ is H_0 -flat of rank μ . In fact, it is clear that $H_0^\wedge[[\underline{x}]]_N/(\frac{\partial F_0}{\partial x_i})$ is H_0^\wedge -flat of rank μ , therefore $H_0[[\underline{x}]]_N/(\frac{\partial F_0}{\partial x_i})$ is H_0 -flat of rank μ independent of $N \geq N_0$, and so $H_0[[\underline{x}]]/(\frac{\partial F_0}{\partial x_i})$ is also H_0 -flat of rank μ .

Assume therefore that $S_0[[\underline{x}]]_N/(\frac{\partial F_S}{\partial x_i})$ is S -flat and consider the diagram above for $N \geq N_0$. Since all horizontal sequences are exact, and since we already know that the extreme left and the extreme right vertical sequences are exact, we find by diagram chasing that f maps $\ker \lambda$ onto itself. But $f \in (\underline{x})$ so by Nakayama $\ker \lambda = 0$, and we are through.

Q.E.D.

Now, (3.7) implies the following, assuming $k = \underline{c}$.

Theorem (4.3). Let f be an isolated singularity with $\mu = \mu(f)$, $\tau_0 = \tau(f)$. Then there exists a finite collection of analytic families

$$\Pi_{\tau, i}: \underline{X}_{\tau, i} \rightarrow \underline{M}_{\tau, i}, \quad 0 \leq \tau \leq \tau_0, \quad i = 1, \dots, m_\tau$$

containing as fibers all of the isolated singularities of the versal family $F = f + \sum_{i=1}^{\tau} \underline{x}^{\alpha_i} t_i$. Moreover, $\underline{M}_{\tau, i}$ is connected,

and $\Pi_{\tau,i}$ is not constant along any nontrivial subscheme of $\underline{M}_{\tau,i}$.

Finally, the μ -constant deformations of f correspond to a sub collection

$$\{\Pi_{\tau,i} \mid (\tau,i) \in \Lambda_\mu, \quad \Lambda_\mu \subseteq \{(\tau,i) \mid 0 < \tau < \tau_0, \quad i=1, \dots, m_\tau\}.$$

Proof. This is just (3.7) together with (4.2).

Q.E.D.

The rest of this § is devoted to the study of deformations of isolated singularities f , with f quasihomogenous of weights $\frac{1}{a_1}, \dots, \frac{1}{a_n}$ and degree 1. The conditions on the weights are not essential. The aim of this § is the computation of the μ -modality of f , i.e. of the integer $\mu(f) = \max_{(\tau,i) \in \Lambda_\mu} \{\dim \underline{M}_{\tau,i}\}$. Our first task is to make explicite the Kodaira-Spencer map, in this local case. More precicesly, we want to compute the kernel \underline{V} of

$$g.: \theta_{\underline{H}} \rightarrow A^1(H., \tilde{A}.; \tilde{A}.)$$

where we have put $\tilde{A}. = (H. [\underline{x}] / (F.))^S$.

Since f is quasihomogenous $\mu(f) = \tau(f)$ and $A^1(k, A.; A.) = k[[x_1, \dots, x_n]] / (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. Therefore $\{\underline{x}^\alpha\}_{\alpha \in I}$,

$I = \{(\alpha_1, \dots, \alpha_n) \mid 0 < \alpha_i < a_i - 2\}$ is a basis for $A^1(k, A.; A.)$. Let

$I. = I \setminus \{(0, \dots, 0)\}$ and let for every $\alpha \in I$, $\deg \alpha = \sum_{i=1}^n \frac{\alpha_i}{a_i}$. Put

$I_\mu = \{\alpha \in I \mid \deg \alpha > 1\}$. Then

$$H. = k[t_\alpha]_{\alpha \in I.}.$$

Corresponding to the inclusion $I_\mu \subset I.$ there is a quotient morphism

$$H. \twoheadrightarrow H_\mu$$

where we have put $H_\mu = k[t_\alpha]_{\alpha \in I_\mu}$. Let g_μ be the restriction of

g. to $\underline{H}_\mu = \text{Spec}(H_\mu)$. Then

$$H_\mu \rightarrow \tilde{A}_\mu$$

where $\tilde{A}_\mu = (H_\mu[\underline{x}] / (F_\mu))_S$, $F_\mu = f + \sum_{\underline{\alpha} \in I_\mu} t_{\underline{\alpha}} \underline{x}^\alpha$, is the versal μ -constant family of isolated singularities defined by f . And

$$g_\mu: \theta_{\underline{H}_\mu} \rightarrow A^1(H_\mu, \tilde{A}_\mu; \tilde{A}_\mu)$$

is defined by

$$g_\mu\left(\frac{\partial}{\partial t_{\underline{\alpha}}}\right) = \frac{\partial F_\mu}{\partial t_{\underline{\alpha}}} = \underline{x}^\alpha, \quad \underline{\alpha} \in I_\mu.$$

Consider the Euler relation E of F_μ ,

$$E = \sum_{\underline{\alpha} \in I_\mu} (\deg_{\underline{\alpha}} - 1) t_{\underline{\alpha}} \underline{x}^\alpha.$$

E defines a map

$$E: H_\mu[[\underline{x}]] / \left(\frac{\partial F_\mu}{\partial x_1}, \dots, \frac{\partial F_\mu}{\partial x_n}\right) \rightarrow H_\mu[[\underline{x}]] / \left(\frac{\partial F_\mu}{\partial x_1}, \dots, \frac{\partial F_\mu}{\partial x_n}\right)$$

by multiplication.

Notice that, by assumption on μ -constancy, $\{\underline{x}^\alpha\}_{\underline{\alpha} \in I}$ is a basis of $H_\mu[[\underline{x}]] / \left(\frac{\partial F_\mu}{\partial x_1}, \dots, \frac{\partial F_\mu}{\partial x_n}\right)$ as H_μ -free-module.

Pick $\underline{\alpha} \in I$, then

$$E \underline{x}^\alpha = \sum_{\underline{\beta} \in I_\mu} h_{\underline{\alpha}\underline{\beta}} \underline{x}^\beta, \quad h_{\underline{\alpha}\underline{\beta}} \in H_\mu.$$

Lemma (4.4). $\underline{V}_\mu = \ker g_\mu$ is generated by the vector fields

$$\delta_{\underline{\alpha}} = \sum_{\underline{\beta} \in I_\mu} h_{\underline{\alpha}\underline{\beta}} \frac{\partial}{\partial t_{\underline{\beta}}}, \quad \underline{\alpha} \in I.$$

Proof. Obviously E is zero in $A^1(H_\mu, \tilde{A}_\mu; \tilde{A}_\mu)$ therefore $\delta_{\underline{\alpha}} \in \underline{V}_\mu$.

Conversely, suppose $\sum_{\beta \in I_\mu} g_\beta \frac{\partial}{\partial t_\beta} \in \underline{V}_\mu$, Then

$$\sum_{\beta \in I_\mu} g_\beta x^\beta = \sum_{i=1}^n p_i \frac{\partial F_\mu}{\partial x_i} + p \cdot F_\mu = \sum_{i=1}^n (p_i + \frac{x_i}{a_i} \cdot p) \frac{\partial F_\mu}{\partial x_i} - p \cdot E. \text{ Now}$$

$$p \cdot E \equiv \sum_{\alpha \in I} h_\alpha \cdot E x^\alpha \text{ for some } h_\alpha \in H_\mu, \text{ modulo } (\frac{\partial F_\mu}{\partial x_1}, \dots, \frac{\partial F_\mu}{\partial x_n}).$$

Since $\{x^\alpha\}_{\alpha \in I}$ is an H_μ -free basis for $H_\mu[[x]] / (\frac{\partial F_\mu}{\partial x_1}, \dots, \frac{\partial F_\mu}{\partial x_n})$ it follows that

$$\sum_{\beta \in I_\mu} g_\beta x^\beta = \sum_{\alpha \in I} h_\alpha \sum_{\beta \in I_\mu} h_{\alpha\beta} x^\beta$$

in $H_\mu[[x]] / (\frac{\partial F_\mu}{\partial x_1}, \dots, \frac{\partial F_\mu}{\partial x_n})$, therefore

$$\sum_{\beta \in I_\mu} g_\beta \frac{\partial}{\partial t_\beta} = \sum_{\alpha \in I} h_\alpha \delta_\alpha$$

$$\text{in } \theta_{H_\mu} = \bigoplus_{\beta \in I_\mu} H_\mu \cdot \frac{\partial}{\partial t_\beta}.$$

Denote by $K = K(\underline{t})$ the matrix $(h_{\underline{\alpha}, \underline{\beta}})_{\underline{\alpha}, \underline{\beta} \in I_\mu}$, then by (3.5) for every point $(\underline{t}) \in H_\mu$

$$\begin{aligned} \tau(F(\underline{t})) &= \mu\text{-rank } K(\underline{t}) \\ \text{sm}(F(\underline{t})) &= \dim\{\tilde{t} \mid \text{rank } K(\tilde{t}) = \text{rank } K(\underline{t})\} \\ &\quad - \text{rank } K(\underline{t}). \\ \mu m(f) &= \max_{\underline{t} \in H_\mu} \text{sm}(F(\underline{t})). \end{aligned}$$

Notice that $\mu m(f)$ is the usual modality of f with respect to the μ -constant stratum, under the action of the contact group. Put

$$\begin{aligned} \tau_{\min.} &= \min_{\underline{t} \in H_\mu} \tau(F(\underline{t})) \\ m_0(f) &= \dim H_\mu, \text{ the inner modality of } f, \end{aligned}$$

then the main results of this paper are,

Theorem (4.5). $\mu m(f) = m_0(f) + \tau_{\min} - \mu.$

Theorem (4.6). (i) $\underline{t} \rightarrow sm(F(\underline{t}))$ is upper semicontinuous.

(ii) $\tau(F(\underline{t})), \underline{t} \in \underline{H}_\mu$ takes every possible value between τ_{\min} and $\tau(f) = \mu.$

Theorem (4.5) is a consequence of (4.6) (i). To prove (4.6) we shall have to study the matrix $K(\underline{t})$ more carefully. Notice first that (4.4) together with (3.5) implies

Proposition (4.7). $H_0 = H_\mu / \mathcal{O}$, where \mathcal{O} is the ideal generated by $t_{\underline{\alpha}}, \deg \underline{\alpha} > 1.$

Proof. We know that $H_0 = H_\mu / (h_{\underline{\alpha}, \underline{\beta}})$, but $h_{\underline{\alpha}, \underline{\beta}}$ is zero for all $\underline{\alpha}, \underline{\beta}$ iff E is zero and $E = \sum_{\underline{\alpha} \in I_\mu} (\deg \underline{\alpha} - 1) t_{\underline{\alpha}} x^{\underline{\alpha}}$ is zero iff $t_{\underline{\alpha}}$ is zero for all $\underline{\alpha}$ with $\deg \underline{\alpha} > 1.$ Q.E.D.

It follows from (4.7) that we may write

$$H_0 = k[t_{\underline{\alpha}}], \underline{\alpha} \in I, \deg \underline{\alpha} = 1.$$

Now let $\lambda_0 < \lambda_1 < \dots < \lambda_\mu$ be the monomial basis $\{x^{\underline{\alpha}}\}_{\underline{\alpha} \in I}$ ordered by degree and lexicographic order, i.e. such that

$$x^{\underline{\alpha}} < x^{\underline{\beta}} \text{ or } \underline{\alpha} < \underline{\beta}$$

iff either $\deg \underline{\alpha} < \deg \underline{\beta}$ or $\deg \underline{\alpha} = \deg \underline{\beta}$ and $\underline{\alpha} = (\underline{\alpha}', \alpha_n), \underline{\beta} = (\underline{\beta}', \beta_n)$ with $\underline{\alpha}' < \underline{\beta}'.$

Remark (4.8). (i) $E \cdot \lambda_i$ is contained in the submodule of

$$H_\mu[[x_1, \dots, x_n]] / \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right) \text{ generated by } \{ \lambda_k \mid \deg \lambda_k > 1 \}$$

- (ii) $\lambda_0 = 1$, $\lambda_\mu = x_1^{a_1-2} \dots x_n^{a_n-2}$
 (iii) duality; $\lambda_i \cdot \lambda_{\mu-i} = \lambda_\mu$. We shall write $\lambda_i^\vee = \lambda_{\mu-i}$.
 (iv) $E\lambda_i = 0$ if $\deg \lambda_i > \deg \lambda_\mu - \min\{\deg \lambda_k \mid \deg \lambda_k > 1\}$.

From this it follows that the matrix K looks like

$$\begin{matrix} & \overbrace{\hspace{1.5cm}}^r \\ r \left\{ \begin{pmatrix} K_0 \\ \hline 0 \end{pmatrix} \right. \\ \mu-r \left\{ \end{matrix} \right., \quad r = m_0(f) - sm(f) = \#\{k \mid \deg \lambda_k > 1\}$$

Let M_1 be the sub H_μ -module of $H_\mu[[x_1, \dots, x_n]] / (\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})$ generated by the λ_i 's with $\deg \lambda_i \leq \deg \lambda_\mu - \min\{\deg \lambda_k \mid \deg \lambda_k > 1\}$, i.e. by the r first λ_i 's, and let M_2 be the sub H_μ -module generated by the λ_i 's with $\deg \lambda_i > 1$, i.e. by those x^α 's sitting strictly above the face of f .

Notice that $\lambda_i \in M_1$ iff $\lambda_i^\vee \in M_2$. Put $\mu_j = \lambda_{r-j}^\vee$ and $t_j = t_{\alpha_j}$; if $\mu_j = x^{\alpha_j}$. Then $\{\lambda_i\}_{i=1}^r$ is a basis for M_1 , and $\{\mu_j\}_{j=1}^r$ is a basis for M_2 .

With these notations we observe that K_0 is the matrix associated to the H_μ -linear map

$$E: M_1 \rightarrow M_2$$

defined by multiplication. Before we proceed with the general theory, let us consider an example.

Example (4.9). Let $f = x^5 + y^{11}$, $n = 2$. Then

$$F_\mu(\underline{t}) = f + t_1 xy^9 + t_2 x^2 y^7 + t_3 x^3 y^5 + t_4 x^2 y^8 + t_5 x^3 y^6 \\ + t_6 x^2 y^9 + t_7 x^3 y^7 + t_8 x^3 y^8 + t_9 x^3 y^9$$

$$E = \frac{1}{55}(t_1 xy^9 + 2t_2 x^2 y^7 + 3t_3 x^3 y^5 + 7t_4 x^2 y^8 + 8t_5 x^3 y^6 + 12t_6 x^2 y^9 \\ + 13t_7 x^3 y^7 + 18t_8 x^3 y^8 + 23t_9 x^3 y^9)$$

This is easily seen by inspecting the Newton diagram of f .

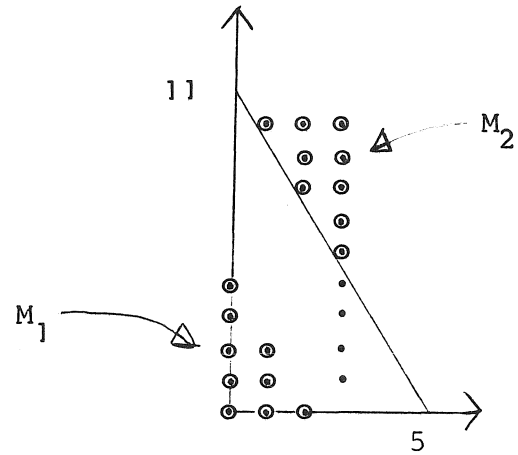
$$a_1 = 5 \quad a_2 = 11$$

$$\mu(f) = \tau(f) = 27$$

$$m_0(f) = 9$$

$$\text{sm}(f) = 0$$

$$r = 9, H_\mu = k[t_1, \dots, t_9]$$



Now, put $A = 2t_2 - \frac{9}{11}t_1^2$, $B = 3t_3 - \frac{7}{11}t_1t_2$, then the matrix $55 \cdot K_0$ looks like:

$$\begin{pmatrix} t_1 & 2t_2 & 3t_3 & 7t_4 & 8t_5 & 12t_6 & 13t_7 & 18t_8 & 23t_9 \\ 0 & 0 & 0 & A & B & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & A & B & * & * \\ 0 & 0 & 0 & 0 & 0 & t_1 & 2t_2 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B - \frac{9}{11}t_1A & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B - \frac{9}{11}t_1A \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 \end{pmatrix}$$

the flattening stratification $\{s_\tau\}_{\tau \in \Lambda_\mu}$ of $A^1(H_\mu, \tilde{A}_\mu; \tilde{A}_\mu)$, see § 3, coincides with the rank-filtration of K_0 , and it is easily seen that $\Lambda_\mu = \{21, 22, 23, 24, 25, 26, 27\}$ is the set of possible Tjurina numbers with constant $\mu (= 27)$ in the neighbourhood of f .

Moreover one calculates, and get

$$\begin{aligned}
S_{21} &= \{\underline{t} \in \underline{H}_\mu \mid \text{Rk } K_0(\underline{t}) = 6\} = \{\underline{t} = (t_1, \dots, t_9) \mid t_1 B - 2t_2 A \neq 0\} \\
S_{22} &= \{\underline{t} \in \underline{H}_\mu \mid \text{Rk } K_0(\underline{t}) = 5\} = \{\underline{t} \mid t_1 B - 2t_2 A = 0, A \neq 0 \text{ or } B \neq 0\} \\
S_{23} &= \{\underline{t} \in \underline{H}_\mu \mid \text{Rk } K_0(\underline{t}) = 4\} = \{\underline{t} \mid t_1 = t_2 = t_3 = 0, t_4 \neq 0 \text{ or } t_5 \neq 0\} \\
S_{24} &= \{\underline{t} \in \underline{H}_\mu \mid \text{Rk } K_0(\underline{t}) = 3\} = \{\underline{t} \mid t_1 = t_2 = t_3 = t_4 = t_5 = 0, t_7 \neq 0\} \\
S_{25} &= \{\underline{t} \in \underline{H}_\mu \mid \text{Rk } K_0(\underline{t}) = 2\} = \{\underline{t} \mid t_1 = t_2 = t_3 = t_4 = t_5 = t_7 = 0, t_6 \neq 0 \text{ or } t_8 \neq 0\} \\
S_{26} &= \{\underline{t} \in \underline{H}_\mu \mid \text{Rk } K_0(\underline{t}) = 1\} = \{\underline{t} \mid t_1 = \dots = t_8 = 0, t_9 \neq 0\} \\
S_{27} &= \{0\}
\end{aligned}$$

and one gets the table

τ	21	22	23	24	25	26	27
$\mu(f)$	3	3	2	1	1	0	0

Thus

$$\mu m(f) = 3.$$

The nice properties of the matrix K_0 becomes apparent if one restrict attention to the linear terms of $K_0(\underline{t})$. The corresponding matrix will be denoted $L(\underline{t})$. In fact $55 \cdot L(\underline{t})$ looks like:

$$\begin{pmatrix}
t_1 & 2t_2 & 3t_3 & 7t_4 & 8t_5 & 12t_6 & 13t_7 & 18t_8 & 23t_9 \\
0 & 0 & 0 & 2t_2 & 3t_3 & 7t_4 & 8t_5 & 13t_7 & 18t_8 \\
0 & 0 & 0 & 0 & 0 & 2t_2 & 3t_3 & 8t_5 & 13t_7 \\
0 & 0 & 0 & 0 & 0 & t_1 & 2t_2 & 7t_4 & 12t_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3t_3 & 8t_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2t_2 & 7t_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3t_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2t_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1
\end{pmatrix}$$

Notice that $L(\underline{t})$ is symmetric on the antidiagonal.

Now it turns out that this symmetry is, in a certain sense, a general fact. So let us return to the case of a quasihomogeneous isolated singularity f with weights $\frac{1}{a_1}, \dots, \frac{1}{a_n}$.

Put $K_0(\underline{t}) = (h_{ij})$ and let $L(\underline{t}) = (\lambda_{ij})$ be the corresponding matrix of the linear forms λ_{ij} of h_{ij} . Recall that

$E = \sum d_i t_i \mu_i$, $d_i = \deg \mu_i - 1$, and that $E\lambda_i = \sum_{j=1}^r h_{ij} \mu_j$. Put

$d_i = \deg \lambda_i - 1$, $i = 1, \dots, r$. Recall also that the purpose of the study of $K_0(\underline{t})$ is to find the flattening filtration $\{S_\tau\}_{\tau \in \Lambda_\mu}$.

With this purpose in mind it is easy to see that we may assume f has the very special form $f(\underline{x}) = \sum_{i=1}^n x_i^{a_i}$. In fact any quasihomogeneous f with weights $\frac{1}{a_1}, \dots, \frac{1}{a_n}$, will after a suitable coordinate change, have the form

$$f(\underline{x}) = \sum_{i=1}^n x_i^{a_i} + \sum_{\deg \underline{\alpha} = 1} k_{\underline{\alpha}} x^{\underline{\alpha}}.$$

Now change coordinates in the prorepresenting substratum

$\underline{H}_0 = \text{Spec}(k[t_{\underline{\alpha}} | \deg \underline{\alpha} = 1])$ to obtain the required form.

Proposition (4.10). With the notations above,

- (i) $\lambda_{ij} = \lambda_{r-j, r-i}$
- (ii) $\lambda_{ij} = d_{k(i, j)} t_{k(i, j)}$ whenever $j > j_i = \min\{j, h_{ij} \neq 0\}$
- (iii) $k(i, j) < k(i, j+1)$ and $k(i+1, j) < k(i, j)$
- (iv) $j_i < j_{i+1}$.

Proof. (i) Since $E\lambda_i = \sum_{j=1}^r h_{ij} \mu_j$ and since $\lambda_{r-\ell} = \sum \mu_\ell$ we find

$$E\lambda_i \lambda_{r-\ell} = \sum_{j < \ell} h_{ij} \mu_j \sum_{\mu_\ell} + h_{i\ell} \mu_r.$$

Since $E\lambda_i \lambda_{r-l} = E\lambda_{r-l} \lambda_i$ and since the linear part of

$\sum_{j < l} h_{ij} \mu_j^\vee \mu_l^\vee$ is of the form $\sum_{s < r} * \mu_s$ it follows that

$$\lambda_{ij} = \lambda_{r-j, r-i}, \text{ i.e. (i).}$$

(ii) and the first part of (iii) is clear. To prove the rest we need the following lemma.

Lemma (4.11). There is a collection of disjoint subsets J_ℓ of

$\{\lambda_1, \dots, \lambda_r\}$, $\ell = 1, \dots, k$, such that

$$(1) \quad \Lambda = \{\lambda_1, \dots, \lambda_r\} = \bigcup_{\ell=D}^k J_\ell \quad \text{and,}$$

$$(2) \quad J_0 = \{\lambda_0\}, J_k = \{\lambda_i \mid 1 < \deg \lambda_i < 1 + \frac{1}{a_j} \text{ for all } j = 1, \dots, n\}$$

$$(3) \quad J_t < J_{t+1}, \text{ i.e. for } \lambda_i \in J_t, \lambda_j \in J_{t+1}, \text{ will } \lambda_i < \lambda_j$$

$$(4) \quad J_t \cdot J_{k-t} \subseteq J_k$$

$$(5) \quad \text{If } \lambda \in \Lambda \text{ and } \{\lambda\} > J_t, \text{ and } \lambda' \in J_{k-t} \text{ then } \{\lambda \cdot \lambda'\} > J_k$$

$$(6) \quad \text{Let } \lambda \in \Lambda \text{ and suppose } \lambda \cdot J_t \subseteq J_k \text{ and } \{\lambda \cdot \lambda'\} > J_k \text{ for all } \lambda' \in \Lambda \text{ with } \{\lambda'\} > I_t, \text{ then } \lambda \in J_{k-t+s} \text{ for some } s > 0.$$

Using this lemma, it is not difficult to see that

$$j_i = \# \left(\bigcup_{v=k-l+1}^k J_v \right) \text{ if } \lambda_i \in J_\ell.$$

This implies (iv) and $k(i+1, j) < k(i, j)$, thus (4.10).

Proof of lemma. Suppose by induction that we have already constructed J_{t-1} and J_{k-t+1} with the required properties. Let $\lambda \in \Lambda$ be minimal among those with $\{\lambda\} > J_{t-1}$, and let $\tilde{\lambda} \in \Lambda$ be maximal among those with $\{\tilde{\lambda}\} < J_{k-t+1}$. Because of (5) and (6) we find $\lambda \cdot \tilde{\lambda} \in J_k$. Let $J_t = \{\lambda' \in \Lambda \mid \lambda' \cdot \tilde{\lambda} \in J_k, \lambda < \lambda'\}$. Choose $\bar{\lambda} \in \Lambda$ minimal among those with $\{\bar{\lambda}\} > J_t$ and put $J_{k-t} = \{\lambda' \in \Lambda \mid \lambda' < \tilde{\lambda}, \lambda \cdot \lambda' \in J_k, \lambda' \cdot \bar{\lambda} > J_k\}$. It is easy to check that $J_t \cdot J_{k-t} \subseteq J_k$ and that (5) and (6) also hold.

Q.E.D.

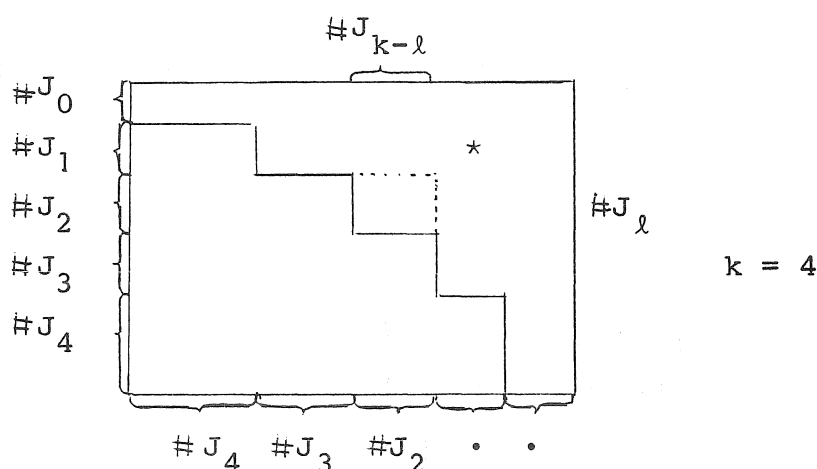
Example (4.12). If $f = x^5 + y^{11}$ we find,

$$\Lambda = \{\lambda_1, \dots, \lambda_9\} = \{y, y^2, x, y^3, xy, y^4, xy^2, x^2\}$$

$$J_0 = \{1\}, J_1 = \{y\}, J_2 = \{y^2, x\}, J_3 = \{y^3, xy\},$$

$$J_4 = \{y^4, xy^2, x^2\}.$$

Notice that K_0 in this case looks like,



Remark (4.13). If we knew that $\#J_i \leq \#J_{k-i}$ for all $0 \leq i \leq k-1$, then

the maximal rank of $L(\underline{t})$ would be $2 \sum_{i < \frac{k}{2}} \#J_i + \#J_{k/2}$ where

$J_{k/2} = \emptyset$ if $k/2 \notin \mathbb{Z}$. In general this is, however, not true.

Proof of (4.6). We first prove the second part, and as a beginning we shall concentrate on the computation of the rank of the linear part $L(\underline{t})$ of $K_0(\underline{t})$.

So let's compute the determinant of the minors of $L(\underline{t})$.

We shall use (4.10). In particular, it follows that any minor of $L(\underline{t})$ has the form,

$$M(\underline{t}) = (d_{a(i,j)} t_{a(i,j)})$$

where $a(i,j) \in \{0, 1, \dots, r\}$, $i, j = 1, \dots, m$, $d_{a(i,j)} = \deg \lambda_{a(i,j)}^{-1} \neq 0$,

if $a(i,j) \neq 0$, $d_0 = 0$, and where

(1) $a(i,j-1) \neq 0$ implies $a(i,j) < a(i,j+1)$

(2) $a(i-1,j) \neq 0$ implies $a(i,j) < a(i-1,j)$.

Lemma (4.14). With the notations above,

$\det M(\underline{t}) \neq 0$ if and only if

$a(i,i) \neq 0$ for all $i = 1, \dots, m$.

Proof. Suppose $a(i,i) = 0$ for some i , then $M(\underline{t})$ has the form

$$i - \begin{pmatrix} & & & & i \\ & & & & \vdots \\ & & & & * \\ & & & & \vdots \\ & & & & * \\ 0 & \dots & 0 & * & \dots & * \\ \vdots & & & 0 & & \\ \vdots & & & 0 & & \\ 0 & \dots & 0 & & & \end{pmatrix}$$

therefore $\det M(\underline{t}) = 0$.

Assume $a(i,i) \neq 0$ for all $i = 1, \dots, m$. Use induction on the number of $a(i,j) \neq 0$. Let $s = \min\{a(i,j) \mid a(i,j) \neq 0\}$, and assume

$$\det M(\underline{t})_{t_s=0} = 0.$$

By induction $M(\underline{t})_{t_s=0}$ has a diagonal element $\bar{a}(i,i) = 0$. This implies that $M(\underline{t})$ has the form

$$i - \begin{pmatrix} & & & & i \\ & & & & \vdots \\ & & & & * \\ & & & & \vdots \\ & & & & * \\ \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} & \begin{pmatrix} d_s & t_s & * & \dots & * \\ * & & & & \\ * & & & & \\ 0 & & & & \end{pmatrix} \end{pmatrix}$$

where A and B are minors of $L(\underline{t})$, and therefore has the same form as $M(\underline{t})$. By induction $\det A \neq 0$, $\det B \neq 0$, therefore

$$\det M(\underline{t}) = d_s \cdot t_s \cdot \det A \cdot \det B \neq 0. \quad \text{Q.E.D.}$$

Now, it is clear that (4.10), (ii) together with (4.14) shows that the rank of $K_0(\underline{t})$ is the same as the rank of $L(\underline{t})$.

Moreover, if we agree to call "a diagonal" any string of elements of a matrix parallel to the diagonal, we may prove the following.

Lemma (4.15). The rank of $L(\underline{t})$ is the length ℓ of the maximal diagonal containing no zeros.

Proof. From (4.14) we deduce $\ell \leq \text{rank } L(\underline{t})$. To prove the inverse inequality, let $M(\underline{t})$ be any $m \times m$ minor of $L(\underline{t})$ gotten by picking the i_1 'th, i_2 'nd, ..., i_m 'th rows and the j_1 'th, j_2 'nd, ..., j_m 'th. Column of $L(\underline{t})$. Using (4.10) and (4.14) we find that $\det M(\underline{t}) \neq 0$ implies that $\ell_{ir} \neq 0$ for $1 \leq i \leq i_m$, $\ell_{i, r-1} \neq 0$ for $1 \leq i \leq i_{m-1}$, ..., $\ell_{i, r-m} \neq 0$ for $1 \leq i \leq i_1$. Therefore there exists a diagonal of $L(\underline{t})$ of length m containing no zeros. Q.E.D.

The second part of (4.6) now follows. In fact let's consider the maximal diagonal of $K_0(\underline{t})$ containing no zeros. Among the $k(i, j)$ for which $t_{k(i, j)}$ occur on the corresponding diagonal of $L(\underline{t})$, let $k(i_\ell, j_\ell)$ $\ell = 1, \dots, p$ be the smallest. On the subset $T_1 = \{x \in H_\mu \mid \forall \ell, h_{i_\ell j_\ell}(x) = 0 \text{ for } 1 \leq i_\ell \leq i, 1 \leq j_\ell \leq j\}$ the rank of $K_0(\underline{t})$ has decreased by 1. The linear part of $K_0(\underline{t})$ restricted to T_1 turns out to be $L(\underline{t})$ restricted to $t_1 = \dots = t_s = 0$ where $k(i_\ell, j_\ell) = s$, $\ell = 1, \dots, p$. We may therefore continue the procedure, thus proving (4.6) (ii).

To prove (4.6) (i), we need the following lemma.

Lemma (4.16). Let $0 \leq p \leq \max_{\underline{t}} \{\text{rk } K_0(\underline{t})\} = p_0$, then

$$\dim \{\underline{t} \in \underline{H} \mid \text{rk } K_0(\underline{t}) \leq p_0 - p\} \leq r - p.$$

It follows from (4.10) that (4.16) is a consequence of the corresponding statement for the linear part $L(\underline{t})$. Therefore (4.16) is proved if we prove

Lemma (4.17). Let $a(i, j) \in \{0, \dots, r\}$, $i, j = 1, \dots, m$ satisfy

$$(1) \quad a(i, j+1) \neq 0 \quad \text{implies} \quad a(i, j) < a(i, j+1)$$

$$(2) \quad a(i-1, j) \neq 0 \quad \text{implies} \quad a(i, j) < a(i-1, j)$$

and let $d_0 = 0$, $d_s = \deg \mu_s - 1$ as above. Consider $M(\underline{t}) = (d_{a(i, j)} t_{a(i, j)})$, and assume $\det M(\underline{t}) \neq 0$. Let I_p be the ideal generated by the $(p+1)$ -minors of $M(\underline{t})$. Then

$$\text{ht } I_p \geq m - p.$$

Proof. Use induction on m and on the number of different $(t_{a(i, j)})$'s involved in the matrix.

Let $s = \min\{a(i, j) \mid a(i, j) \neq 0\}$ and let U be a component of $V(I_p) = \{\underline{t} \in \underline{H} \mid \forall p \in I_p, p(\underline{t}) = 0\}$.

1. case. $U \subseteq V(t_s) = \{\underline{t} \mid t_s = 0\}$. Consider the sub-matrix $(d_{a(i, j)} t_{a(i, j)})_{1 \leq i \leq m-1, 2 \leq j \leq m}$ obtained from $M(\underline{t})$ by deleting the m^{th} row and the 1st column. The conditions (1) and (2) together with the assumption $\det(d_{a(i, j)} t_{a(i, j)}) \neq 0$ imply that $s \neq a(i, i+1)$ $i = 1, \dots, m-1$. Therefore $\det(d_{a(i, j)} t_{a(i, j)})_{1 \leq i \leq m-1, 2 \leq j \leq m} \neq 0$, and we may apply the induction hypotheses.

2. case. $U \not\subseteq V(t_s)$. It follows from (1) and (2) that $\text{rk } M(\underline{t}) \geq \ell$ for all \underline{t} with $t_s \neq 0$, where ℓ is the number of times t_s

occurs in $M(\underline{t})$, see fig. In particular this implies $\ell \leq p$. Consider the $(m-\ell) \times (m-\ell)$ sub-matrix $M_0(\underline{t})$ of $M(\underline{t})$ obtained by deleting the rows and columns in which t_s occurs.

$$M(\underline{t}) = \begin{pmatrix} * & & & & & & & & & & \\ t_s & * & & & & & & & & & \\ 0 & * & * & * & & & & & & & \\ 0 & 0 & 0 & t_s & * & & & & & & \\ & & 0 & 0 & * & & & & & & \\ & & & 0 & * & * & & & & & \\ & & & 0 & 0 & t_s & * & & & & \\ & & & & 0 & * & & & & & \\ & & & & 0 & * & & & & & \\ & & & & 0 & * & * & * & * & * & * \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & t_s * \\ & & & & & & & & 0 & * & \\ & & & & & & & & 0 & * & \\ & & & & & & & & 0 & * & * \\ & & & & & & & & 0 & 0 & t_s * \end{pmatrix}$$

For $t_s \neq 0$ it is easy to see that $\text{rk } M(\underline{t}) = \ell + \text{rk } M_0(\underline{t})$, therefore

$$\{t \mid \text{rk } M(t) \leq p\} = \{\underline{t} \mid \text{rk } M_0(\underline{t}) \leq p - \ell\}.$$

Note that since $\det M(\underline{t}) \neq 0$, all t_s occurring in $M(\underline{t})$ sit under the diagonal, thus $\det M_0(\underline{t}) \neq 0$. Moreover $M_0(\underline{t})$ does not contain t_s . We may therefore apply the induction hypotheses, and the lemma is proved. Q.E.D.

This ends the proof of theorem (4.6).

In general it seems to be difficult to obtain good formulas for $\mu_m(f)$ depending only on the weights of f . However, in the case of curves we have some partial results.

Proposition (4.18). Let $f = x^a + y^b$ then $\mu(f) = (a-1)(b-1)$.

Moreover, we have

(i) For $b = ra$ the maximal rank of $K(\underline{t})$ is

$$\begin{array}{ll} \frac{1}{4}(a-2)(b-4) & a \text{ even} \\ \frac{1}{4}(a-1)(b-r-4) & a \text{ odd and } r \geq 2 \\ \frac{1}{4}(a-3)^2 & a \text{ odd and } r=1 \end{array}$$

(ii) For $b = ra+1$ or $b = ra-1$ the maximal rank is

$$\begin{array}{ll} \frac{1}{4}(a-2)(b-3) & a \text{ even} \\ \frac{1}{4}(a-1)(b-r-3) & a \text{ odd} \end{array}$$

(iii) For $b = ra+2$ the maximal rank is

$$\begin{array}{ll} \frac{1}{4}(a-3)(b+r-2)+r-1 & a \text{ odd} \\ \frac{1}{4}(a-2)(b+2r-2)+2r-1 & a \text{ even} \end{array}$$

(iv) For $b = ra-2$ the maximal rank is

$$\begin{array}{ll} \frac{1}{4}(a-2)(b-2)-1 & a \text{ even} \\ \frac{1}{4}(a-1)(b-r-2) & a \text{ odd} \end{array}$$

Proof. Will not be given here.

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